# Autosimilar melodies and their implementation in *OpenMusic*

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***Abstract*— Autosimilar melodies put together the interplay of several melodies within a monody, the augmentations of a given melody, and the notion of autosimilarity which is well known in fractal objects. From a mathematical study of their properties, arising from experimentations by their inventor, composer Tom Johnson, we have generalized the notion towards the dual aspects of the group of symmetries of a periodic melody, and the creation of a melody featuring a set of given symmetries. This is now a straightforward tool, both for composers and analysts, in *OpenMusic* visual programming language.**

## I. Introduction

### A. History

Autosimilar melodies have been conceptualized and intensively used by composer Tom Johnson and several american fellows from the 1980's. They were first rigorously defined in the last chapter of his book [5] under the label 'selfRep melodies' (the word 'autoSimilar' being used in a much fuzzier sense in the whole book). We will restrict the meaning of 'autosimilar' to the notion developed thereafter, because it is closer to the common mathematical usage. Also, while keeping close to this traditional meaning, it will be generalized much further than Johnson used it, to musical objects invariant under the action of a given subgroup of the affine automorphisms of some cyclic ring $\mathbf{Z}_n$. This lends itself particularly well to implementation in *OpenMusic*. It must be stressed that examples of autoSimilar melodies crop up in many different musical styles, from Mozart's classical music to New Orleans Jazz. Also the degree of control that our mathematical work entails into the software is a contrapuntist's dream, enabling subtle interplay between a melody and itself at several different tempos.

### B. Notations

$\mathbf{Z}_n$ is the cyclic group with $n$ elements.
We denote the greatest common divisor of $a, n$ by $\gcd(a, n)$.
The invertible elements of $(\mathbf{Z}_n, \times)$ are the generators of the additive group $(\mathbf{Z}_n, +)$; they form a multiplicative group, $\mathbf{Z}_n^*$.
Any set might be given by the list of its elements between curly brackets: $\{0, 3, 5\}$; or by some defining property, e.g. $\mathbf{Z}_n^* = \{a \in \mathbf{Z}_n,\ \gcd(a, n) = 1\}$.
The subgroup generated by some element $g$ of a group $G$ is denoted by $< g >$. e.g. $< a >= (\mathbf{Z}_n, +) \iff a \in \mathbf{Z}_n^*$.

A **periodic melody** $M$ is a map from $\mathbf{Z}_n$ into some musical space, usually pitches or notes, or equivalently a periodic sequence: $\forall k \in \mathbf{Z}, M_{k+n} = M_k$.
The **order** of an element $g$ of a group $G$, denoted by $o(g)$, is the cardinality of $< g >$, i.e. the smallest integer $r > 0$ with $g^r = e$, the unit element of group $G$. It is classically characterised by the following equivalence:

$$g^k = e \iff o(g) \text{ is a divisor of } k$$

## II. Definitions and examples

### A. The original definition

*Definition 1:* A melody with period $n$ is *autosimilar* with ratio $a$ if, taking one note of the melody every $a$ beats, one hears the same melody. Equivalently, it means that the augmentation with ratio $a$ of the original melody is part of it.

### B. Historical examples

The most famous autosimilar melody is probably the Alberti Bass, such as is heard in the first bars of Mozart's Sonata in C major K. 545 (see Fig. 1).

Fig. 1. Alberti Bass with augmentation

Picking out one note every 3 (or 5, or 7, or 9) eighth gives the same melody.

Another example, strikingly different in style, is the thema of Glenn Miller's *In the Mood* (Fig. 2). There the interplay of strong binary beats with the three-periodic melody lets hear the autosimilarity with ratio 4.

Fig. 2. *In the Mood*, measure 14

Voluntary use of autosimilarity is of course plainer in modern pieces, like Johnson's *Loops for Orchestra*, *Kientsy Loops*, or *la Vie est Si Courte* (Fig. 3).


Fig. 3. *La Vie Est Si Courte*

### C. Construction via orbits of an affine map

All the mathematical statements in this presentation have been proved, but we will omit the proofs here for the sake of brevity.

*1) Building an autosimilar melody:*

*Theorem 1:* Any autosimilar melody of ratio $a$ and period $n$ is built up from orbits of the affine map $x \mapsto a \times x \pmod{n}$: if

$$\mathcal{O}_x = \{a^k x \pmod{n}, k \in \mathbf{Z}\} = a^{\mathbf{Z}}.x$$

then for each note $P$ of the melody, the subset of indexes $M^{-1}(P) = \{i \in \mathbf{Z}_n, M_i = P\}$ is one such orbit, or an union of several ones.

This means that one has first to compute the orbits, which are subsets of beat indexes, and choose a single note for all the beats of each orbit – or several orbits.

For example, the orbits of $x \mapsto 3x \mod 8$ are (0), (4), (1 3), (5 7), (2 6) as for instance $0\times3 = 0, 4\times3 = 12 = 4 \mod 8, 5 \times 3 = 7 \mod 8$ and $7 \times 3 = 5$. Setting note C on the first two orbits, i.e. 0 and 4, E on the last one, i.e. 2 and 6, and G on the remaining indexes, one gets the Alberti Bass. This is easily done in *OpenMusic* (see Fig. 4).

*2) About numbers:* Some predictibility about the lenghts of orbits, their total number, its maximal value, and other interesting figures, have been obtained. Let it be observed on this simple example the following general facts:

1) Several orbit lengths are possible;
2) All orbit lengths divide the longest one.
3) The longest orbit has for length $o(a)$.
4) Length 1 (a lone note) occurs;

*3) Prime cases:* Tom Johnson has been particularly interested in periods which are a power of 2, or a prime number. The first case led to the elucidation of the question of the greatest possible number of different notes:

*Theorem 2:* The greatest number of notes for an autosimilar melody with period $n$ is $3n/4$. It occurs when $n$ is a multiple of 4 and $a = 1 + n/2$.

In the prime case on the other hand, all orbits but one are the same size:

*Theorem 3:* If $n$ is prime then $\{0\}$ is one orbit; all other orbits have the same length, $o(a)$.

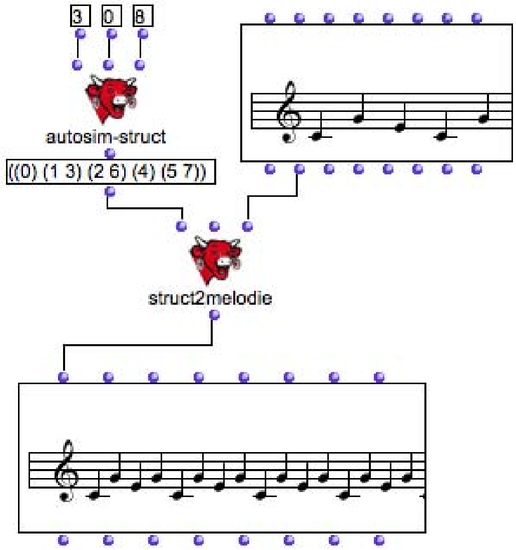


Fig. 4. A patch in *OpenMusic* showing the process leading to the construction of the Alberti Bass

This case can be seen as a realization of a tiling with a motif $\mathcal{O}_1 = \{1, a, a^2 \ldots$ and some augmentations $\mathcal{O}_x = \{x, a\,x, a^2\,x \ldots\} = x \times \mathcal{O}_1$. Fig. 5 shows the tiling of the line process with 0 (which is left aside) and orbit (1 2 4) together with its augmentation (3 6 12). Modulo 7, this reduces of course to the three orbits of $x \mapsto 2x$, namely (0), (1 2 4) and (3 5 6) but musically the effect is different.

This is precious, as few algorithms producing tilings by augmentations are known as yet. *OpenMusic* features another one.

### D. First generalization

The family of maps $x \mapsto ax$ (with $a$ invertible $\mod n$) form a subgroup, $\mathcal{H}_n$, of the general affine group: $\mathrm{Aff}_n = \{x \mapsto a\,x = b, a \in \mathbf{Z}_n^*, b \in \mathbf{Z}_n\}$. The group $\mathcal{H}_n$ of homotheties modulo $n$ is a quotient of $\mathrm{Aff}_n$ by its normal subgroup $\mathcal{T}_n$ of all translations $\tau_b : x \mapsto x + b$, and is isomorphic to the multiplicative group $\mathbf{Z}_n^*$.

Leaving aside this theory, it is obvious to generalize the above definition to such general affine automorphisms:

*Definition 2:* A melody with period $n$ is autosimilar with ratio $a$ and offset $b$ if, taking one note of the melody every $a$ beats, and *starting with the $b^{th}$ beat*, one hears the same melody. Equivalently, it means that the augmentation with ratio $a$ of the original melody is part of it, though maybe with a different starting point.

The construction is identical, giving the same note to all beats belonging to a same orbit of map $f : x \mapsto a\,x + b \mod n$.

More than half the time, this construction gives exactly the same melodies as before: it is sufficient to change the (arbitrary) origin of beats. For instance, the orbits of $x \mapsto 3\,x + 2 \mod 8$ are (0 2), (1 5), (3), (4 6) and (7), which are exactly the same as the orbits of $x \mapsto 3\,x$, up to a translation of 3. In the remaining cases however, new,

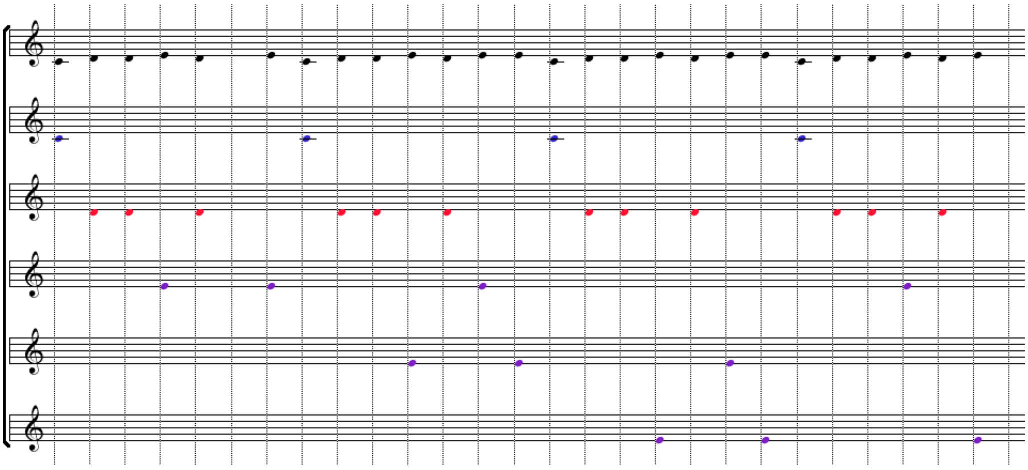

Fig. 5. Tiling the line with one orbit of an autosimilar melody and some of its augmentations

interesting autosimilar melodies are obtained. They have no single notes:

*Theorem 4:* A melody can be generated by $x \mapsto a\,x$, up to some appropriate choice of time origin, iff it admits 1 note orbits.

This new species of autosimilar melodies arising from purely mathematical considerations opens up a wider choice for the composer. It it still true that all different orbit lengths divide the longest one, though its length is now $o(f)$, usually a multiple of $o(a)$.

Example: the popular melodic pattern in figure 6 is autosimilar with ratio 3 and offset 1.

Fig. 6. AutoSimilar with offset

## III. Symmetry group

### A. Stabilizer

*Definition 3:* Let $M = (M_k)_{k \in \mathbf{Z}}$ be an $n-$periodic melody. The set $G_M$ of all $f \in \mathrm{Aff}_n$ verifying

$$\forall k \in \mathbf{Z} \; M_{f(k)} = M_k$$

is a subgroup of $\mathrm{Aff}_n$, called the symmetry group, or stabilizer, of $M$.

This general concept encapsulates the previous definitions: $M$ is autosimilar with ratio $a$, offset $b$, iff the map $f : x \mapsto a\,x + b$ is in its stabilizer. So are *ipso facto* all powers of $f$, and sometimes other maps too: for the Alberti Bass, no fewer than 8 symmetries occur:

$$x \mapsto x, 3x, 5x, 7x, x+4, 3x+4, 5x+4, 7x+4$$

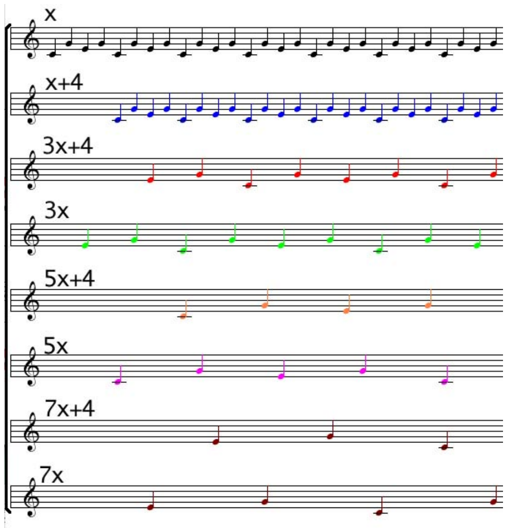


Fig. 7. The family of symmetries for the Alberti Bass

### B. The ultimate definition

It is from there a natural step to define

*Definition 4:* A melody $M$, $n-$periodic, is autosimilar under the subgroup $G$ of $\mathrm{Aff}_n$, iff $G \subset G_M$.

*Theorem 5:* Such a melody can be built by giving the same notes to all elements of a same orbit of group $G$: such an orbit is $\mathcal{O}_x = \{f(x), f \in G\}$.

### C. Algorithms

We have two algorithms about stabilizers implemented in *OpenMusic*:

1) Find the stabilizer of a given melody (with a given period). This is done by checking exhaustively the action of all affine maps in $\mathrm{Aff}_n$ – there are less than $n^2$ such maps. Fig. 7 shows how *OpenMusic* provides the collection of affine transformations for the Alberti Bass, and exhibits their effects on the original melody. This enables the composer to visualize at a glance all the copies of a melody that are present inside itself, i.e. its whole autosimilar potential.

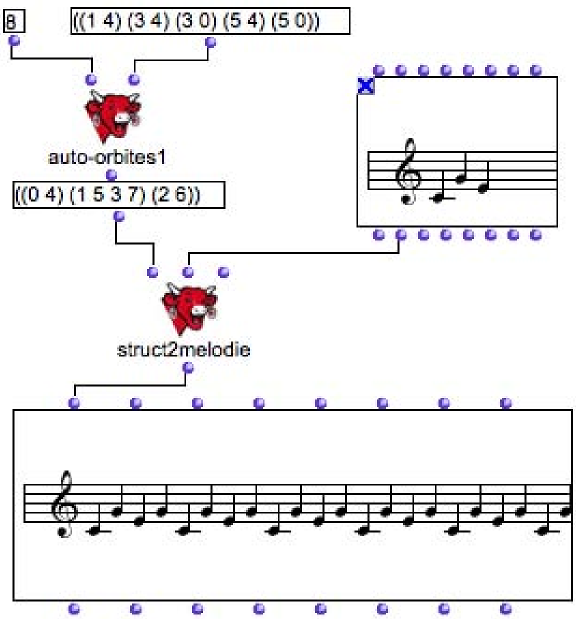


Fig. 8. This patch shows how to produce an autosimilar melody (the Alberti Bass) starting with a set of symmetries, a period and a collection of pitches.

2) Find a melody with some given symmetries. One builds the orbits first as explained below, from there it is the composer's choice to associate notes to each orbit with a standard *OpenMusic* procedure.
The user inputs a collection of affine maps. Starting from an element $x$ in $\mathbf{Z}_n$, a set is initialized with $x$ as sole element. Then all the maps in the collection are applied repeatedly to that set until it no longer changes. All elements of this set, now an orbit, are set aside and the algorithm carries on with the next element in $\mathbf{Z}_n$ that has not been reached yet, until $\mathbf{Z}_n$ is exhausted (see Fig. 8). This is the dual approach from the last one, providing the composer with the simplest structure admitting autosimilar copies with the desired ratios and offsets.

### D. Palindromes

The above concept enables to clarify which autosimilar melodies will be palindromes, as it is only a question of whether $x \mapsto -x$ (or some more general inversion $x \mapsto c - x$) is present in the stabilizer of the melody. The algorithms allow the straighforward construction of palindromic melodies (among other symmetries), and the theory reaches interesting result, as (simplifying a little)

*Theorem 6:* An autosimilar melody with ratio $a$ will be palindromic iff there is some power of $a$ equal to -1 $\mod n$.
For instance, as in Fig. 9 it is clear that $x \mapsto 3\,x$ gives a palindrome modulo 14, as $3^3 = 27 = -1 \mod 14$.

The sequence of periods $n$ owning some such roots of $-1$, that is to say of periods allowing palindromic autosimilar melodies, was previsouly unknown and has been added to Sloane's online encyclopedia of integer

Fig. 9. A palindrome with period 14

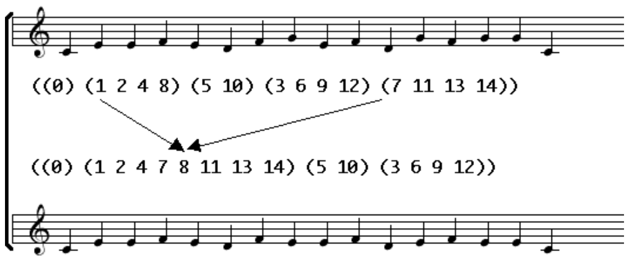


Fig. 10. Autosimilar melody with period 15 and its palindromic deformation

sequences[1] under reference A126949.

Besides of course, it is always possible to build a palindromic (autosimilar) melody from *any* autosimilar melody by just collapsing together notes belonging to orbits that are symmetrical (the map $f : x \mapsto -x$ exchanges orbits of a primitive autosimilar melody with ratio $a$). For example see the original and the 'palindromized' on Fig. 10 obtained by collapsing together the two inversionally-related orbits $(1, 2, 4, 8)$ and $(7, 11, 13, 14)$.[2]

A nice theoretical property of autosimilar melodies appears when one tries to iterate an affine map with a ratio that is *not* invertible modulo $n$: the map being no longer one-to-one, there is no reversibility and some information is lost at each iteration, but (this is related to the very deep Fitting Lemma of commutative algebra that already appeared in a musical context in Anatol Vieru's sequences, [3]) after a time an autosimilar melody emerges (see Fig. 11).

Thus chaos hides inside itself the deepest harmony.

Fig. 11. An autosimilar melody from a random one

## IV. Conclusion

We presented some theoretical and implementational aspects of melodic autosimilarity. After describing a general group-theoretical framework for the construction of autosimilar melodies (via the orbits of an affine transformation), we presented a new approach that generalizes

[1] `http://www.research.att.com/ njas/sequences/`
[2] The remaining orbits are invariant under inversion, e.g. $f(3, 6, 9, 12) = 15 - (3, 6, 9, 12) = (12, 9, 6, 3)$.

Tom Johnson's original definition of self-similar melodies by considering autosimilarity between melodic pattern having different starting point. We then focused on the two main algorithmic aspects of this approach as it has been implemented in *OpenMusic* visual programming language. This implementation provides the set of affine maps that fix a given autosimilar melody and conversely it constructs an autosimilar melody of a given period starting with a family of affine transformations and a collection of pitches. After briefly describing the case of palindromic autosimilar melodies, we concluded the paper by suggesting how to use a general result of commutative algebra in order to establish a possible connection between two apparently very different compositional processes: the construction of autosimilar melodies and Anatol Vieru's theory of finite difference calculus applied on periodic sequences.

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